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Canonical forms of tensor representations and spontaneous symmetry breaking

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Abstract. An algorithm for constructing canonical forms for any tensor representation of the classical compact Lie groups is given. This method is used to find a complete list of the symmetry breaking patterns produced by Higgs fields in the third-rank antisymmetric representations of U(n), SU(n) and SO(n) for $n \leqslant 7$. A simple canonical form is also given for kth-rank symmetric tensor representations.

1. Introduction

Spontaneous symmetry breaking in quantum field theory is often described by the Higgs mechanism. In this formalism a set of scalar fields ϕ transform as some representation ν_{G} of the gauge group G (a compact Lie group). From these fields the Higgs potential $V(\phi)$ is constructed which is required to have the following properties:

(i) $V(\phi)$ is a polynomial in the components of ϕ of degree at most four,

(ii) $\forall g \in \mathrm{G}$, $V(g \cdot \phi) = V(\phi)$, so $V(\phi)$ is G invariant,

(iii) $V(\phi)$ is bounded from below,

(iv) $V(\phi)$ has a local maximum at $\phi = 0$.

The tree-level broken symmetry group is then the little group of the absolute minimum of $V(\phi)$.

This mechanism has been used extensively in grand unified theories to produce the desired pattern of symmetry breaking,

$$\mathrm{G} \rightarrow \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \rightarrow \mathrm{SU}(3) \times \mathrm{U}(1) \tag{1.1}$$

or possibly some variation involving intermediate stages. Despite this fact there are few explicit results for the possible symmetry breaking patterns even in the case when ν_{G} is irreducible. The vector, adjoint and second-rank tensor representations were dealt with by Li (1974) and third-rank antisymmetric tensors have been discussed by Cummins and King (1984) and Jetzer *et al* (1984). In addition certain representations have been investigated in a search for a counterexample to Michel's conjecture (Michel 1979). In particular the 75 of SU(5) (Abud *et al* 1984, Cummins and King 1985, Hubsch *et al* 1984) and the 27 of SU(3) (Burzlaff *et al* 1985) have received attention. Other results included the SU(3) $\times$ U(1) minimum of the 45 of SU(5) by Eckert *et al* (1983).

In the work of both Li (1974) and Burzlaff *et al* (1985) use was made of canonical forms for the representations under consideration. This considerably reduced the calculation involved in minimising the potentials. In this paper a method of constructing simple canonical forms for any tensor representation is given in § 2, and this method is then applied in § 3 to find all the symmetry breaking patterns produced by Higgs fields in the third-rank totally antisymmetric representations of U(n), SU(n) and SO(n) for $n \leqslant 7$. A proof of the algorithm is provided in an appendix.

2. Canonical forms

The canonical forms of the vector, adjoint and second-rank representations of U(n), SU(n) and SO(n) are well known and are given for completeness in table 1. Using these forms Li (1974) found all the possible symmetry breaking patterns produced by these representations, and it thus seems natural to seek a generalisation of these forms to other representations.

To simplify the following results the notation of Abud and Sartori (1983) is adopted, and the reader is referred to this paper for further details. Note though that any representation of a compact Lie group may be realised as a real orthogonal matrix representation acting on $\mathbb{R}^n$ for some suitable n.

The inner product on $\mathbb{R}^n$ will be denoted by

$$\langle \phi, \psi \rangle = \sum_i \phi_i \psi_i, \qquad \phi, \psi \in \mathbb{R}^n \tag{2.1}$$

and the action of G on $\mathbb{R}^n$ will be written

$$g \cdot \phi, \qquad g \in \mathrm{G} \qquad \phi \in \mathbb{R}^n \tag{2.2}$$

Table 1. The canonical forms for the fundamental, second-rank, adjoint and kth-rank symmetric tensor representations of U(n), SU(n) and SO(n). We have used the notation of S functions to specify the representations; for more details see Wybourne (1970) or Cummins and King (1986).

Group	Representation	Dimension	Canonical form
U(n)	$\{1\}$	n	$\phi_1 \in \mathbb{R}$ $\phi_i = 0$ otherwise
	$\{1^2\}$	$\frac{1}{2}n(n-1)$	$\phi_{2i+1,2i+2} \in \mathbb{R}$ $\;0 \leqslant i \leqslant [\frac{1}{2}n-1]$ $\phi_{ij} = 0$ otherwise
	$\{2\}$	$\frac{1}{2}n(n+1)$	$\phi_{ii} \in \mathbb{R}$ $\;0 \leqslant i \leqslant n$ $\phi_{ij} = 0$ otherwise
	$\{\bar{1}; 1\}$	n^2-1	$\phi_i^i \in \mathbb{R}$ $\;0 \leqslant i \leqslant n$ $\phi_i^j = 0$ otherwise
	$\{k\}$	$\dfrac{(n+k-1)!}{k!(n-1)!}$	$\phi_{ii\ldots i} \in \mathbb{R}$ $\phi_{ii\ldots ij} = 0$ $\;i<j$ $\phi_{ij\ldots k} \in C$ otherwise
SU(n)	$\{1\}$	n	as U(n)
	$\{1^2\}$	$\frac{1}{2}n(n-1)$	as U(n), except if n even when $\phi_{12} \in C$
	$\{2\}$	$\frac{1}{2}n(n+1)$	as U(n), except $\phi_{11} \in C$
	$\{\bar{1}; 1\}$	n^2-1	as U(n)
	$\{k\}$	$\dfrac{(n+k-1)!}{k!(n-1)!}$	as U(n), except $\phi_{nn\ldots n} \in C$
SO(n)	$[1]$	n	as U(n)
	$[1^2]$	$\frac{1}{2}n(n-1)$	as U(n)
	$[2]$	$\frac{1}{2}n(n+1)-1$	as U(n) and $\sum \phi_{ii} = 0$

and since the action of G is orthogonal,

$$\langle g \cdot \phi, \psi \rangle = \langle \phi, g^{-1} \cdot \psi \rangle. \tag{2.3}$$

The elements of $\mathscr{L}(\mathrm{G})$, the Lie algebra of G, are $n \times n$ antisymmetric matrices and the action of $\mathscr{L}(\mathrm{G})$ on $\mathbb{R}^n$ will also be written

$$t \cdot \phi, \qquad t \in \mathscr{L}(\mathrm{G}) \qquad \phi \in \mathbb{R}^n. \tag{2.4}$$

The following rather general definition of a canonical form will be used.

Definition 1. Let V be a representation space for a compact Lie group G, and V_c be a subspace of V. If V_c intersects every orbit of the action of G on V at least once, and if V_c contains no proper subspace with this property, then V_c is a canonical form for the action of G on V. In other words any vector in V may be 'rotated' to lie in V_c.

It is well known that the normal space to any point

$$N(\phi) = \{\chi \in \mathbb{R}^n \,|\, (\chi, \mathscr{L}(\mathrm{G}) \cdot \phi) = 0\} \tag{2.5}$$

intersects every orbit of G. The normal space to any point on the generic stratum has the minimal dimension property, and so is a canonical form. Unfortunately, in general, this method of choosing V_c produces a complicated set of constraints, and there is also the problem of finding a suitable point on the generic stratum.

These difficulties may be overcome by systematically removing the freedom associated with the G transformations using the following algorithm.

(i) Choose a vector $\phi^1 \in V$. This vector will eventually form part of the basis for V_c, and typically ϕ^1 will be chosen to have one non-vanishing component. Note that the length of ϕ^1 is arbitrary.

(ii) Find the generators of G that annihilate ϕ^1. Call the group they generate G^1.

(iii) Eliminate all vectors in V of the form $t \cdot \phi^1$, $t \in \mathscr{L}(\mathrm{G})$. Call the remaining space N^1.

(iv) Choose $\phi^2 \in N^1$.

(v) Find the generators of G^1 that annihilate ϕ^2. Call the group they generate G^2.

(vi) Eliminate from N^1 all components of the form $t \cdot \phi^2$, $t \in \mathscr{L}(\mathrm{G}^1)$. Call the remaining space N^2.

(vii) Return to step (iv) (with a suitable relabelling of 1 and 2).

This process is repeated until all possible choices of ϕ^{k+1} are annihilated by $\mathscr{L}(\mathrm{G}^k)$.

For a proof that this algorithm yields a canonical form see the appendix. Note also that G^k is the identity component of a generic little group.

As a trivial example consider the symmetric second-rank tensor representation of SO(3).

(i) Choose ϕ^1 to have non-zero component ϕ_{11}.

(ii) Using the notation $\binom{i}{j}^n_m = \delta^i_m \delta^n_j$, $\mathscr{L}(\mathrm{G}^1)$ is generated by $\binom{2}{3} - \binom{3}{2}$.

(iii) ϕ_{12} and ϕ_{13} are eliminated by the generators $\binom{1}{2} - \binom{2}{1}$ and $\binom{1}{3} - \binom{3}{1}$ respectively.

(iv) Choose ϕ^2 to have non-zero component ϕ_{22}.

(v) $\mathscr{L}(\mathrm{G}^2)$ is trivial.

(vi) ϕ_{23} is eliminated by $\binom{2}{3} - \binom{3}{2}$.

(vii) Since all the generators have been used the algorithm ends here.

Thus all off-diagonal terms have been eliminated, and the standard canonical form for this case is obtained. Note though that because of the arbitrariness in choosing the ϕ^i non-standard canonical forms can also be found if necessary.

The case of complex representations may be included by writing them in a real form (Abud and Sartori 1983), and in this way all the standard forms listed in table 1 may be reproduced. It is also possible to find a very simple canonical form for the kth-rank totally symmetric tensor representation of $U(n)$ and $SU(n)$. This form is included in table 1.

In the case of any other tensor representations it does not seem to be possible to find forms that are valid for all values of n, but it is of course possible to use the algorithm to find a form in any particular case. The forms for totally antisymmetric third-rank tensor representations are given in table 2 for $n \leq 7$, and these will be used in the next section.

3. Application to symmetry breaking produced by third-rank antisymmetric tensors

It was shown in Cummins and King (1984) that sufficient conditions for ϕ_{ijk} to be an absolute minimum of

$$V(\phi) = -\tfrac{1}{2}\mu^2 \sum_{ijk} \phi_{ijk}\phi^{ijk} + \tfrac{1}{4}\lambda_1 \left(\sum_{ijk} \phi_{ijk}\phi^{ijk} \right)^2 + \tfrac{1}{4}\lambda_2 \sum_{\substack{ijk \\ pql}} \phi_{ijk}\phi^{ijl}\phi_{pql}\phi^{pqk} \tag{3.1}$$

where $(\phi_{ijk})^* = \phi^{ijk}$ are

(I) if $-3\lambda_1 < \lambda_2 < 0$, then (after a suitable transformation) only one component of ϕ is non-zero, say ϕ_{123};

(II) if $\lambda_2 > 0$ and $n\lambda_1 + \lambda_2 > 0$, then $\Sigma_{ij}\ \phi_{ijl}\phi^{ijk} \propto \delta_l^k$.

These conditions do not fix $|\phi|$ which must be determined by substitution into (3.1). In general for other values of λ_1 and λ_2 the potential is not bounded below.

In case (I) the symmetry breaking pattern is easy to compute. In case (II) the constraint may be solved using the canonical forms of table 2. In fact for $n = 4, 5$ the constraint has no solutions, but these cases are easy to solve explicitly. The results of all these calculations are shown in table 3.

Table 2. The canonical forms for third-rank totally antisymmetric tensors for $U(n)$, $SU(n)$ and $SO(n)$, $n \leq 7$.

n	Canonical form		$U(n)$	$SU(n)$	$SO(n)$
4	$\phi_{123} = a$		$a \in \mathbb{R}$	$a \in \mathbb{R}$	$a \in \mathbb{R}$
5	$\phi_{125} = a$				
	$\phi_{345} = b$		$a, b \in \mathbb{R}$	$a, b \in \mathbb{R}$	$a, b \in \mathbb{R}$
6	$\phi_{123} = a$	$\phi_{246} = d$	$a, b, c, d, e \in \mathbb{R}$	$a, c, d \in \mathbb{R}$	$a, \ldots, e \in \mathbb{R}$
	$\phi_{345} = b$	$\phi_{456} = e$		$b, e \in C$	
	$\phi_{156} = c$				
7	$\phi_{123} = a$	$\phi_{356} = h$	$a, b, c, d, e, g, h \in \mathbb{R}$	$a, b, d, e, g, h \in \mathbb{R}$	$a, \ldots, h \in \mathbb{R}$
	$\phi_{146} = b$	$\phi_{357} = i$	$f, i, j, k, l, m, n \in C$	$c, f, i, j, k, l, m, n \in C$	
	$\phi_{157} = c$	$\phi_{367} = j$			
	$\phi_{167} = d$	$\phi_{456} = k$			
	$\phi_{245} = e$	$\phi_{457} = l$			
	$\phi_{267} = f$	$\phi_{467} = m$			
	$\phi_{347} = g$	$\phi_{567} = n$			

Table 3. The symmetry breaking patterns produced by the Higgs potential for third-rank totally antisymmetric tensors. Note that the cases $n = 4, 5$ do not satisfy the constraint of case (II) (and consequently the boundedness condition is less strict in these cases), but they are included here for convenience.

(*a*) *Case* (II) *for* U(n).

n	Little algebra	Embedding in U(n)	Example of minimising vector	
4	SU(3)+U(1)	$\{1\}\to\{1\}_0+\{0\}_{-1}$	$\phi_{123}\neq 0$	
5	Sp(4)+U(1)	$\{1\}\to\langle 1\rangle_1+\langle 0\rangle_{-2}$	$\phi_{125}:\phi_{345}$ 1 : 1	
6	SU(3)+SU(3)	$\{1\}\to\{1\}\times\{0\}+\{0\}\times\{1\}$	$\phi_{123}:\phi_{456}$ 1 : 1	
7	G_2	$\{1\}\to(1, 0)$	123 : 145 : 167 : 246 : 357	$1:1:1:\sqrt{2}:\sqrt{2}$

To obtain the SU(n) case, delete the U(1) factor.

(*b*) *Case* (II) *for* SO(n).

n	Little algebra	Embedding in SO(n)
4	SU(2)	$[1]\to\{1\}+\{0\}$
5	SU(2)+U(1)	$[1]\to\{1\}_1+\{1\}_{-1}$
6	SU(2)+SU(2)	$[1]\to\{1\}\times\{0\}+\{0\}\times\{1\}$
7	G_2	$[1]\to(1, 0)$

The examples of minimising vectors for SO(n) may be taken to be the same as those for U(n).

(*c*) *Case* (I).

Group	Little algebra	Embedding
U_n	SU(3)+U($n-3$)	$\{1\}\times\{0\}+\{0\}\times\{1\}$
SU(n)	SU(3)+SU($n-3$)	$\{1\}\times\{0\}+\{0\}\times\{1\}$
SO(n)	SO(3)+SO($n-3$)	$[1]\times[0]+[0]\times[1]$

It should perhaps be stressed that we have calculated the little algebra (the Lie algebra of the little group) for the various minima using the condition

$$(t\cdot\phi)_{ijk}=\sum_m (t_i^m\phi_{mjk}+t_j^m\phi_{imk}+t_k^m\phi_{ijm})=0 \tag{3.2}$$

for $t\in\mathscr{L}(G_\phi)$. In some cases, however, the little group may not be connected (Abud *et al* 1984, Burzlaff *et al* 1985, Cummins and King 1986).

The results are in agreement with Jetzer *et al* (1984) and Cummins and King (1984); they also establish the *uniqueness* of the symmetry breaking patterns.

For $n\geqslant 8$ the constraint (II) becomes too complicated to solve by hand, although some progress could probably be made using algebraic computing.

For $n=8$ one possible solution to (II) is given by choosing $\phi_{ijk}=f_{ijk}$, the structure constants of SU(3). This produces the pattern

$$G\to SU(3) \tag{3.3}$$

where G = U(8), SU(8) or SO(8). Since in each case condition (II) contains more constraints (albeit non-linear) than degrees of freedom in the canonical form it seems reasonable to conjecture that (3.3) is the unique symmetry breaking pattern for this case. When $n\geqslant 9$, however, it is possible to show that the solutions of (II) do not yield unique symmetry breaking patterns.

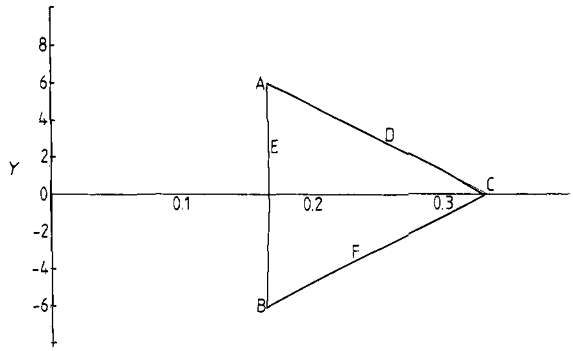

Figure 1. The boundary of the region Ω defined by equations (3.6) and (3.7). The corresponding vectors are given by: A, $\phi_{123}:\phi_{456}$ 1 : 1; B, $\phi_{123}:\phi_{456}$ 1 : 1; C, $\phi_{123}\neq 0$; D, $\phi_{123}:\phi_{456}$ $1:t$; E, $\phi_{123}:\phi_{456}$ $1:\exp\frac{1}{2}\mathrm{i}\pi t$; F, $\phi_{123}:\phi_{456}$ $1:\mathrm{i}t$; where $0<t<1$. In all cases the little algebra of $\partial\Omega$ is SU(3) + SU(3).

In the case of SU(6) it is not in fact true that (3.1) is the most general Higgs potential. This is due to the existence of the additional fourth-order invariant

$$W(\phi)=\mathrm{Re}\left(\lambda_3\sum_{\substack{a,b,c\\ l,m,n}}\tilde{\phi}^{lmn}\tilde{\phi}^{abc}\phi_{lma}\phi_{nbc}\right) \tag{3.4}$$

where

$$\tilde{\phi}^{lmn}=\sum_{p,q,r}\varepsilon^{lmnpqr}\phi_{pqr} \tag{3.5}$$

and

$$\lambda_3\in C$$

Furthermore the conditions (I) and (II) do not apply when this term is included in the potential.

To find the symmetry breaking pattern in this case first note that by redefining ϕ we may choose λ_3 to be real. The problem now reduces to finding the absolute minimum of a potential with three quartic and no cubic terms. This type of minimisation has a simple geometric interpretation (Kim 1982) in terms of finding the 'most peripheral' parts of the boundary of the closed bounded subset, Ω, of $\mathbb{R}^2$ defined by

$$X=\sum\phi_{ijk}\phi^{ijl}\phi_{pql}\phi^{pqk}\left(\sum\phi_{ijk}\phi^{ijk}\right)^{-2} \tag{3.6}$$

$$Y=\mathrm{Re}\left(\sum\tilde{\phi}^{lmn}\tilde{\phi}^{abc}\phi_{lma}\phi_{nbc}\right)\left(\sum\phi_{ijk}\phi^{ijk}\right)^{-2} \tag{3.7}$$

Using the canonical form for SU(6), X and Y depend on only seven real parameters so that it is not difficult to determine Ω by choosing many random values for these parameters and plotting the results using a computer. The result is shown in figure 1. Somewhat surprisingly the symmetry breaking patterns are the same as those when (3.3) is absent, namely SU(6) $\rightarrow$ SU(3) + SU(3).

4. Conclusion

In § 2 a method has been given for constructing simple canonical forms of tensor representations of compact Lie groups. These forms simplify the explicit minimisation of Higgs potentials and this is illustrated in § 3 where the forms for third-rank antisymmetric tensors are used to establish unique symmetry breaking patterns for U(n), SU(n) and SO(n) for $n \leq 7$.

It should be noted that there are, of course, many other examples of cases which involve tensors transforming as representations other than vector, second-rank or adjoint, and the construction of simple canonical forms should prove useful for explicit calculations in these cases.

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Appendix

In this appendix we give a proof that the algorithm of § 2 does indeed give a canonical form. First consider any sequence $\{\phi^k\}$ where $\phi^k \in \mathbb{R}^n \forall k \in Z$ and $\phi^0 = 0$.

Definition 2. $G_{\phi k} = \{g \in \mathrm{G} \mid g \cdot \phi^k = \phi^k\}$ (the little group of ϕ^k).

Definition 3. Define G^k by
(i) $\mathrm{G}^0 = \mathrm{G}$
(ii) $\mathrm{G}^k = \mathrm{G}_{\phi^k} \cap \mathrm{G}^{k-1}$.

Definition 4. Define N^k by
(i) $N^0 = \mathbb{R}^n$
(ii) $N^k = \{\phi \in N^{k-1} \mid (\phi, \mathscr{L}(\mathrm{G}^{k-1}) \cdot \phi^k) = 0\}$.

It will be seen that these definitions are in agreement with the notation used in the algorithm. In order to show that N^k intersects all orbits the following two lemmas are needed

Lemma 1. $\forall k$, $\mathrm{G}^k \cdot N^k \subset N^k$

Proof by induction. If $\phi \in N^k \subset N^{k-1}$, and $g \in \mathrm{G}^k \subset \mathrm{G}^{k-1}$, then

$$(g \cdot \phi, \mathscr{L}(\mathrm{G}^{k-1})\phi^k) = (\phi, g^{-1}\mathscr{L}(\mathrm{G}^{k-1})g\phi^k) = (\phi, \mathscr{L}(\mathrm{G}^{k-1})\phi^k) = 0 \tag{A1}$$

and also by the inductive hypothesis

$$g \cdot \phi \in \mathrm{G}^{k-1} \cdot N^{k-1} \subset N^{k-1}. \tag{A2}$$

So from definition 4, using (A1) and (A2)

$$g \cdot \phi \in N^k. \tag{A3}$$

Clearly the case $k = 0$ is trivial, and the lemma follows.

Lemma 1 means that N^k is a representation space for G^k, and so G^k orbits on N^k are well defined. This fact we need to show is the following.

Lemma 2. If $\Omega^k \subset N^k$ is a G^k orbit on N^k then $\Omega^k \cap N^{k+1} \neq \varnothing$.

Proof (following Abud and Sartori). Let

$$d\langle \Omega^k, \phi^{k+1}\rangle = \inf_{\phi \in \Omega^k} |\phi - \phi^{k+1}|^2. \tag{A4}$$

Ω^k is compact, and hence there exists $\chi \in \Omega^k$ such that

$$d\langle \Omega^k, \phi^{k+1}\rangle = |\chi - \phi^{k+1}|^2. \tag{A5}$$

Consider f: $\mathrm{G}^k \to \mathbb{R}$ defined by

$$\begin{aligned} f(g) &= 2(g \cdot \chi, \phi^{k+1}) \\ &= |g \cdot \chi - \phi^{k+1}|^2 + |\chi|^2 + |\phi^{k+1}|^2 \\ &= -d\langle g \cdot \chi, \phi^{k+1}\rangle + \text{constant}. \end{aligned} \tag{A6}$$

By the definition of $\chi, f(g)$ has a maximum at $g = e$ so that

$$\begin{aligned} 0 &= (\mathscr{L}(\mathrm{G}^k)\chi, \phi^{k+1}) \\ &= (\chi, \mathscr{L}(\mathrm{G}^k)\phi^{k+1}). \end{aligned} \tag{A7}$$

Hence from definition 3, $\chi \in N^{k+1}$ and so $N^{k+1} \cap \Omega^k \neq \varnothing$.

Using this lemma it is easy to show the following.

Proposition 1. If Ω is a G orbit on $\mathbb{R}^n$, then $\Omega \cap N^k \neq \varnothing\ \forall k$.

Proof by induction. By hypothesis we may choose $\Psi \in \Omega \cap N^{k-1}$, and from lemma 2 we may choose $\rho \in \Omega^{k-1}(\Psi) \cap N^k$. But clearly $\rho = h \cdot \Psi$ for some $h \in \mathrm{G}^{k-1} \subset \mathrm{G}$, and so $\rho \in \Omega(\Psi) \cap N^k = \Omega \cap N^k$.

The case $k = 0$ is trivial and so the proposition follows.

All that remains to be shown is that the final N^k satisfies the minimum dimension requirement. To show this we first need the following lemma.

Lemma 3. $\dim N^k = \dim V - \dim \mathrm{G} + \dim \mathrm{G}^k\ \forall k$.

Proof. It is well known that for any compact Lie group H acting on a representation space W, the following is true:

$$\forall \Psi \in W, \qquad \dim N(\Psi) = \dim W - \dim \mathrm{H} + \dim \mathrm{H}_\psi. \tag{A8}$$

Applying this result to the action of G^{k-1} on N^{k-1}, using lemma 1 and the assumption that $\phi^k \in N^{k-1}$ we have

$$\begin{aligned} \dim N^k &= \dim N^{k-1} - \dim \mathrm{G}^{k-1} + \dim \mathrm{G}^{k-1} \cap \mathrm{G}_{\phi^k} \\ &= \dim N^{k-1} - \dim \mathrm{G}^{k-1} + \dim \mathrm{G}^k \end{aligned} \tag{A9}$$

and the lemma follows by induction.

Now let d be the dimension of a generic little group and q be the dimension of a generic normal space (which is also the dimension of the orbit space). It may be shown that

$$\dim N^k \geqslant q. \tag{A10}$$

If it is also the case that G^k leaves N^k pointwise fixed, then since N^k contains generic points

$$\dim G^k \leqslant d \tag{A11}$$

and so using (A8)

$$\dim G^k \leqslant q - \dim V + \dim G$$

and from lemma 3

$$\dim N^k \leqslant q$$

so that

$$\dim N^k = q. \tag{A12}$$

Thus we have shown the following.

Proposition 2. The algorithm produces a canonical form.

It also follows immediately that

$$\dim G^k = d \tag{A13}$$

so that G^k is the identity component of some generic little group.

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